ELEN E3401: Electromagnetics

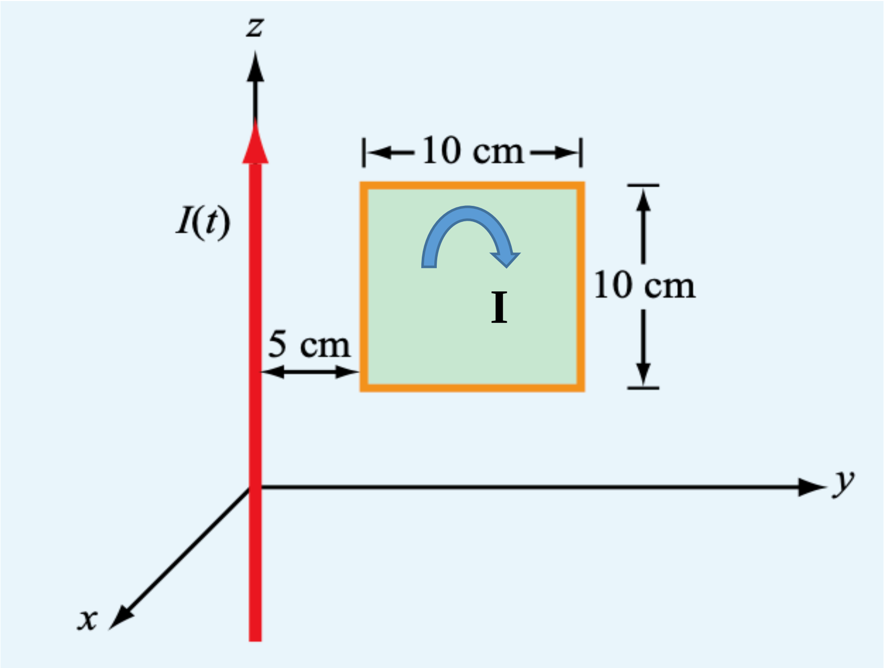
Spring 2025

Prof. Keren Bergman

Exam 2 Review

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Example 1

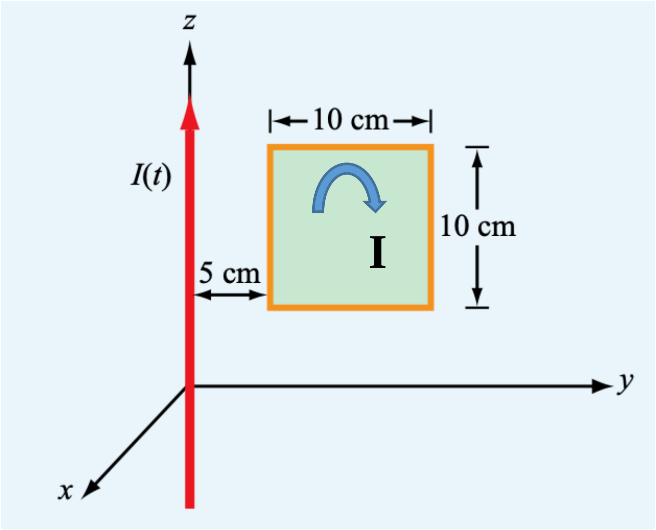


Loop is coplanar with long, straight wire carrying current, along *z*-axis:

$$I(t) = 5\cos(2\pi \times 10^4 t) \quad [A]$$

a) Determine V_{emf} induced across 5cm gap

Example 1



Loop is coplanar with long, straight wire carrying current, along z-axis:

$$I(t) = 5\cos(2\pi \times 10^4 t) \quad [A]$$

a) Determine V_{emf} induced across 5cm gap

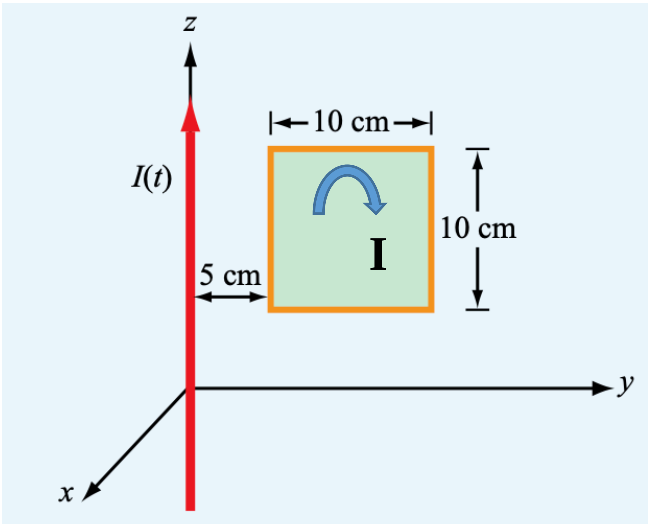
Plan to get V_{emf} :

obtain the flux: $\Phi = \int_S \vec{B} \cdot d\vec{s}$

then, get: $V_{emf}^{tr} = -\dfrac{d\Phi}{dt}$

To obtain the flux, we obtain magnetic flux density due to wire: $\vec{B} = \hat{\varphi}\dfrac{\mu_0 I}{2\pi r} = -\hat{x}\dfrac{\mu_0 I}{2\pi y}$

Example 1



Loop is coplanar with long, straight wire carrying current, along z-axis:

$$I(t) = 5\cos(2\pi \times 10^4 t) \quad [A]$$

a) Determine V_{emf} induced across 5cm gap

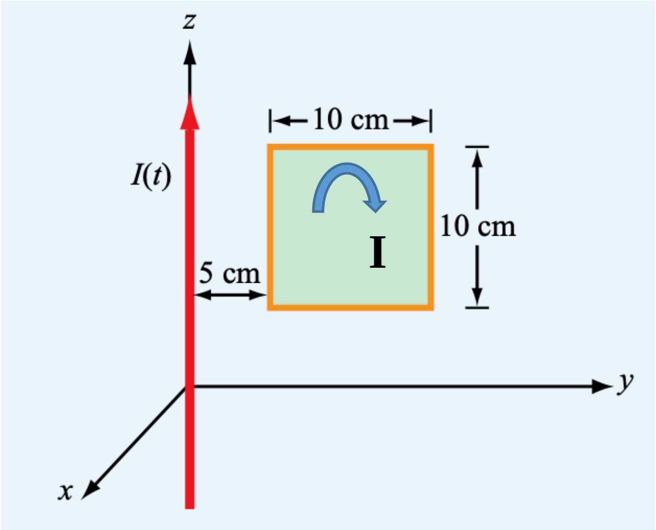
$$\vec{B} = \hat{\varphi}\frac{\mu_0 I}{2\pi r} = -\hat{x}\frac{\mu_0 I}{2\pi y}$$

In the plane of the loop: $\hat{\varphi} = -\hat{x} \qquad r = y$

Flux passing through loop: $$\Phi = \int_S \vec{B} \cdot d\vec{s} = \int_{5cm}^{15cm} \left(-\hat{x}\frac{\mu_0 I}{2\pi y}\right) \cdot (-\hat{x}\ 10cm)dy = \frac{\mu_0 I \times 0.1}{2\pi} \ln\left(\frac{15}{5}\right)$$

$$\Phi = \frac{(4\pi \times 10^{-7})(5\cos(2\pi \times 10^4 t)) \times 0.1}{2\pi} \times 1.1 = 1.1 \times 10^{-7}\cos(2\pi \times 10^4 t) \quad [Wb]$$

Example 1



Loop is coplanar with long, straight wire carrying current, along *z*-axis:

$$I(t) = 5\cos(2\pi \times 10^4 t) \quad [A]$$

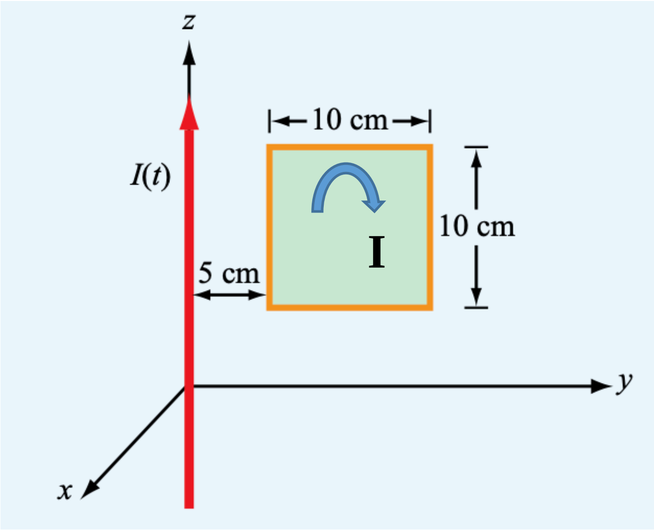
a) Determine V_{emf} induced across 5cm gap

$$V_{emf}^{tr} = -\frac{d\Phi}{dt}$$

$$V_{emf}^{tr} = -\frac{d\Phi}{dt} = (1.1)(2\pi \times 10^4)\sin(2\pi \times 10^4 t) \times 10^{-7}$$

$$V_{emf}^{tr} \;\; = 6.9 \times 10^{-3} \sin(2\pi \times 10^4 t) \; [V]$$

Example 1



b) Obtain current flowing through a 4Ω resistor connected across gap

Loop has internal resistance of 1 Ω.

$$I_{ind} = \frac{V_{emf}^{tr}}{4+1} = \frac{6.9 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 1.38 \sin(2\pi \times 10^4 t) \quad [mA]$$

Direction of induced current: $\vec{B} = -\hat{x}\dfrac{\mu_0 5\cos(2\pi \times 10^4 t)}{2\pi y}$

At $t = 0$, $\vec{B}$ is max and points in $-\hat{x}$. As time increases, $\vec{B}$ will decrease. So the <u>induced current is CW</u> – creates increasing $\vec{B}$ in $-\hat{x}$

Propagation in lossy medium: any medium

Table 7-1

$$\alpha = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} - 1 \right] \right\}^{1/2}$$

$$\beta = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} + 1 \right] \right\}^{1/2}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon'}} \left(1 - j\frac{\epsilon''}{\epsilon'} \right)^{-1/2}$$

$$u_p = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} \qquad \epsilon' = \epsilon \qquad \epsilon'' = \frac{\sigma}{\omega}$$

Propagation in lossy medium: approximations

1. Lossless $\quad \alpha = 0 \qquad \beta = \omega\sqrt{\mu\epsilon}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon}} \qquad u_p = \frac{1}{\sqrt{\mu\epsilon}} \qquad \lambda = \frac{u_p}{f}$$

2. Low-loss: $\quad \frac{\epsilon''}{\epsilon'} \ll 1 \qquad \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \qquad \beta = \omega\sqrt{\mu\epsilon}$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon}} \qquad u_p = \frac{1}{\sqrt{\mu\epsilon}} \qquad \lambda = \frac{u_p}{f}$$

2. Good conductor: $\quad \frac{\epsilon''}{\epsilon'} \gg 1 \qquad \alpha = \sqrt{\pi f \mu\sigma} \qquad \beta = \sqrt{\pi f \mu\sigma}$

$$\eta_c = (1+j)\frac{\alpha}{\sigma} \qquad u_p = \sqrt{\frac{4\pi f}{\mu\sigma}} \qquad \lambda = \frac{u_p}{f}$$

Propagation in lossy medium: any medium

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1 + j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

Example 2

Animal tissue: $\epsilon_r = 12, \mu_r = 1, \sigma = 0.3 \left[\frac{S}{m}\right]$ $at\ f = 100MHz$

Is it a low loss dielectric or good conductor?

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{0.3}{(2\pi \times 10^8)12 \times 8.85 \times 10^{-12}} = 4.5 \leftarrow \text{Neither!}$$

Have to use full equations for α, β, η_c

$$\alpha = 9.75 \left[\frac{Np}{m}\right] \qquad \beta = 12.16 \left[\frac{rad}{m}\right] \qquad \lambda = \frac{2\pi}{\beta} = 0.52\ m$$

$$u_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^8}{12.16} = 0.52 \times 10^8 \left[\frac{m}{s}\right]$$

$$\eta_c = \sqrt{\frac{\mu_0}{12\epsilon_0}}(1 - j4.5)^{-1/2} = 39.54 + j31.72\ \Omega$$

Example 3

3GHz wave with amplitude 10 V/m is incident on wet soil (ignore reflection)

Wet soil: $\epsilon_r = 9, \mu_r = 1, \sigma = 5 \times 10^{-4} \left[\frac{S}{m}\right]$

At what depth will amplitude reduce to 1 mV/m?

$$E(z) = E_0 e^{-\alpha z} = 10e^{-\alpha z}$$

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{5 \times 10^{-4}}{2\pi \times 3 \times 10^9 \times 9 \times 8.85 \times 10^{-12}} = 3.32 \times 10^{-4} \rightarrow low\ loss\ dielectric$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{9\epsilon_0}} = 0.032$$

$$\frac{1mV}{m} = 0.001 = 10e^{-0.032z}$$

$$\ln(10^{-4}) = -0.032z$$

$$\Rightarrow z = 287.82\ m$$

Example 4

$$\tilde{E}^i = \hat{y}8\cos(6\pi \times 10^9 t - 30\pi x) \left[\frac{V}{m}\right]$$

Media 1 and 2 are non-conducting

Medium 1 | Medium 2

Plane wave →

$\epsilon_{r_1} = 2.25, \mu_r = 1$ | $\epsilon_{r_2} = 4, \mu_r = 1$

non conducting → no losses → $\eta \rightarrow$ real

a) Obtain $\vec{E}_1(x,t), \vec{E}_2(x,t), \vec{H}_1(x,t), \vec{H}_2(x,t)$

First steps – obtain the reflection/transmission coefficients

$$\eta_1 = \frac{\eta_0}{\sqrt{\epsilon_{r1}}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33\ \Omega \qquad \eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r2}}} = \frac{377}{\sqrt{4}} = 188.5\ \Omega$$

$$\Gamma = \frac{(\eta_2 - \eta_1)}{(\eta_2 + \eta_1)} = \frac{\left(\frac{1}{\sqrt{\epsilon_{r2}}} - \frac{1}{\sqrt{\epsilon_{r1}}}\right)}{\frac{1}{\sqrt{\epsilon_{r2}}} + \frac{1}{\sqrt{\epsilon_{r1}}}} = \frac{\left(\frac{1}{2} - \frac{1}{1.5}\right)}{\frac{1}{2} + \frac{1}{1.5}} = -0.143$$

$$\tau = 1 + \Gamma = 1 - 0.143 = 0.857$$

Example 4

Medium 1 | Medium 2

Plane wave

$\epsilon_{r_1} = 2.25, \mu_r = 1$

$\epsilon_{r_2} = 4, \mu_r = 1$

Next steps – obtain fields in medium 1

$$\vec{E}^r = \Gamma \vec{E}^i = -1.14\hat{y}\cos(6\pi \times 10^9 t + 30\pi x)$$

$$\vec{E}_1 = \vec{E}^r + \vec{E}^i = \hat{y}[8\cos(6\pi \times 10^9 t - 30\pi x) - 1.14\cos(6\pi \times 10^9 t + 30\pi x)]$$

$$\vec{H}^i = \hat{z}\frac{8}{\eta_1}\cos(6\pi \times 10^9 t - 30\pi x) = \hat{z}31.83\cos(6\pi \times 10^9 t - 30\pi x) \quad \left[\frac{mA}{m}\right]$$

$$\tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E}$$

$$\vec{H}^r = \hat{z}\frac{1.14}{\eta_1}\cos(6\pi \times 10^9 t + 30\pi x)$$

$$\vec{H}_1 = \vec{H}^r + \vec{H}^i = \hat{z}[31.83\cos(6\pi \times 10^9 t - 30\pi x) + 4.54\cos(6\pi \times 10^9 t + 30\pi x)]$$

Example 4

Next steps – obtain fields in medium 2

Medium 1 | Medium 2

Plane wave $\rightarrow$

$\epsilon_{r_1} = 2.25, \mu_r = 1$ | $\epsilon_{r_2} = 4, \mu_r = 1$

$$k_1 = \omega\sqrt{\mu\epsilon_1} \qquad k_2 = \omega\sqrt{\mu\epsilon_2}$$

$$k_2 = \sqrt{\frac{\epsilon_2}{\epsilon_1}}k_1 = \sqrt{\frac{4}{2.25}}30\pi = 40\pi\left[\frac{rad}{m}\right]$$

$$\vec{E}_2 = \vec{E}^t = \hat{y}8\,\tau\cos(6\pi \times 10^9 t - 40\pi x) = \hat{y}6.86\cos(6\pi \times 10^9 t - 40\pi x)$$

$$\vec{H}_2 = \vec{H}^t = \hat{z}8\frac{\tau}{\eta_2}\cos(6\pi \times 10^9 t - 40\pi x) = \hat{z}36.38\cos(6\pi \times 10^9 t - 40\pi x)$$

Example 4

b) Obtain the average incident/reflected and transmitted power densities:

Avg power densities:

$$\vec{S}^{i}_{av} = \hat{x}\frac{|E^{i}_{0}|^2}{2\eta_1} = \frac{\hat{x}64}{2(251.33)} = \hat{x}127.3 \ \left[\frac{mW}{m^2}\right]$$

$$\left|E^{i}_{0}\right| = 8$$

$$\vec{S}_{av} = \frac{1}{2}Re\left[\tilde{E}_1 \times \widetilde{H}^{*}_{1}\right] \qquad \vec{S}_{av_1} = \hat{k}\frac{|E^{i}_{0}|^2}{2\eta_1}(1 - |\Gamma|^2)$$

$$\vec{S}^{r}_{av} = -|\Gamma|^2\vec{S}^{i}_{av} = -\hat{x}(0.143)^2(0.127) = -\hat{x}2.6 \ \left[\frac{mW}{m^2}\right]$$

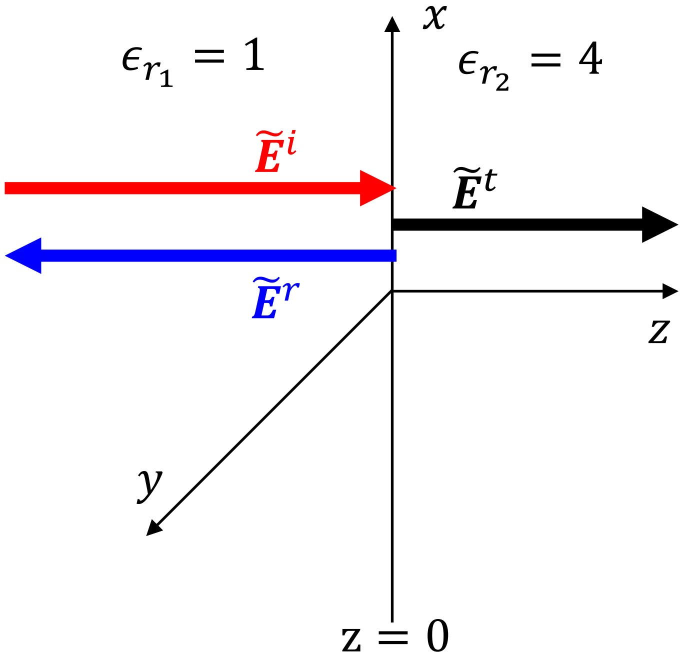
$$\vec{S}^{t}_{av} = \hat{x}\frac{|E^{t}_{0}|^2}{2\eta_2} = \hat{x}\frac{\tau^2(8)^2}{2\eta_2} = \hat{x}\frac{0.86^2(8)^2}{2(188.5)} = \hat{x}124.7 \ \left[\frac{mW}{m^2}\right]$$

$$\vec{S}^{i}_{av} + \vec{S}^{r}_{av} = \vec{S}^{t}_{av}$$

Example 5

A 200 MHz, left-hand circularly (LHC) polarized plane wave with an electric field modulus (magnitude) of 5 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 4$, and occupies the region defined by $z \geq 0$

a) Write an expression for the electrical field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$



$$k_1 = \frac{\omega}{c} = \frac{2\pi \times 2 \times 10^8}{3 \times 10^8} = \frac{4\pi}{3} \left[\frac{rad}{m}\right]$$

$$k_2 = \frac{\omega}{c}\sqrt{\epsilon_{r_2}} = \frac{4\pi}{3}\sqrt{4} = \frac{8\pi}{3} \left[\frac{rad}{m}\right]$$

LHC wave:

$$\widetilde{\boldsymbol{E}}^i = a_0\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{-jkz}$$

$$\boldsymbol{E}^i(z,t) = \hat{x}a_0\cos(\omega t - kz) - \hat{y}a_0\sin(\omega t - kz)$$

Example 5

A 200 MHz, left-hand circularly (LHC) polarized plane wave with an electric field modulus (magnitude) of 5 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 4$, and occupies the region defined by $z \geq 0$

a) Write an expression for the electrical field phasor of the incident wave, given that the field is a positive maximum at $z = 0$ and $t = 0$

LHC wave:

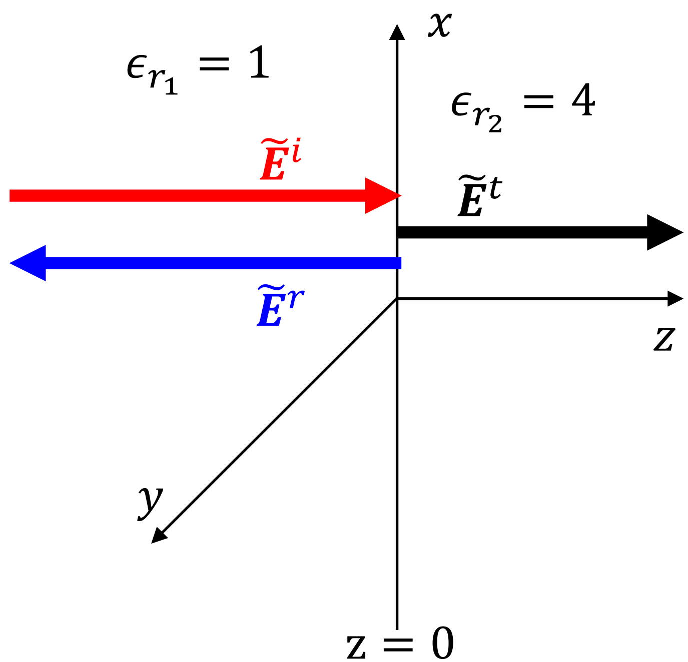
$$\tilde{\boldsymbol{E}}^i = a_0\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{-jkz}$$

$$\boldsymbol{E}^i(z,t) = \hat{x}a_0\cos(\omega t - kz) - \hat{y}a_0\sin(\omega t - kz)$$

$$\left|\boldsymbol{E}^i\right| = \left[\hat{x}a_0^2\cos^2(\omega t - kz) + \hat{y}a_0^2\sin^2(\omega t - kz)\right]^{1/2}$$

$$\left|\boldsymbol{E}^i\right| = a_0 = 5\left[\frac{V}{m}\right]$$

$$\tilde{\boldsymbol{E}}^i = 5\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{-j\frac{4\pi}{3}z}\left[\frac{V}{m}\right]$$



Example 5

A 200 MHz, left-hand circularly polarized plane wave with an electric field modulus (magnitude) of 5 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 4$, and occupies the region defined by z ≥ 0

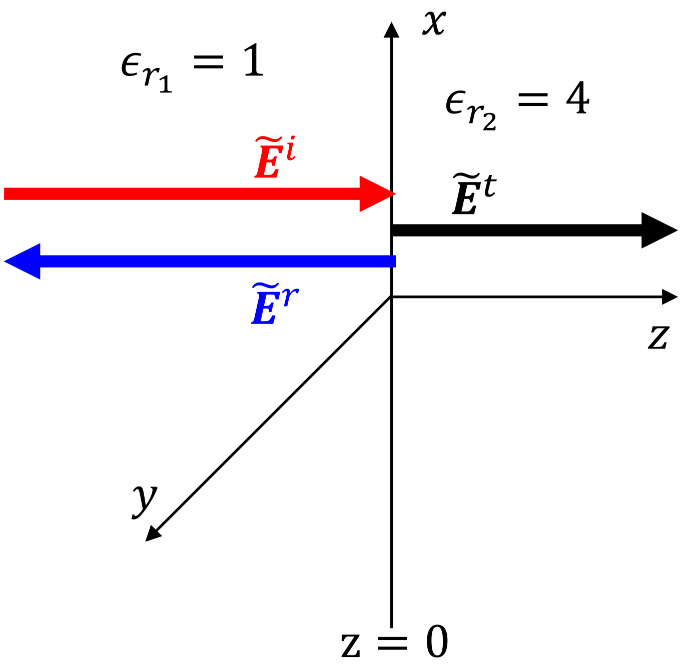
b) Calculate the reflection and transmission coefficients

$$\eta_1 = \eta_0 = 120\pi \;\; [\Omega]$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r_2}}} = \frac{\eta_0}{2} = 60\pi \;\; [\Omega]$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = \frac{-60}{180} = -\frac{1}{3}$$

$$\tau = 1 + \Gamma = \frac{2}{3}$$



Example 5

A 200 MHz, left-hand circularly polarized plane wave with an electric field modulus (magnitude) of 5 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 4$, and occupies the region defined by z ≥ 0

c) Write the expressions for the electric field phasors of the reflected wave, the transmitted wave, and the total field in the region z ≤ 0

$$\widetilde{\boldsymbol{E}}^r = 5\Gamma\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{j\frac{4\pi}{3}z} = -\frac{5}{3}\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{j\frac{4\pi}{3}z}\left[\frac{V}{m}\right]$$

$$\widetilde{\boldsymbol{E}}^t = 5\tau\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{-j\frac{8\pi}{3}z} = \frac{10}{3}\left(\hat{x} + \hat{y}e^{j\pi/2}\right)e^{-j\frac{8\pi}{3}z}\left[\frac{V}{m}\right]$$

$$\widetilde{\boldsymbol{E}}_1 = \widetilde{\boldsymbol{E}}^i + \widetilde{\boldsymbol{E}}^r = 5\left(\hat{x} + \hat{y}e^{j\pi/2}\right)\left[e^{-j\frac{4\pi}{3}z} - \frac{1}{3}e^{j\frac{4\pi}{3}z}\right]\left[\frac{V}{m}\right]$$

Example 5

A 200 MHz, left-hand circularly polarized plane wave with an electric field modulus (magnitude) of 5 V/m is normally incident in air upon a dielectric medium with $\epsilon_r = 4$, and occupies the region defined by z $\geq$ 0

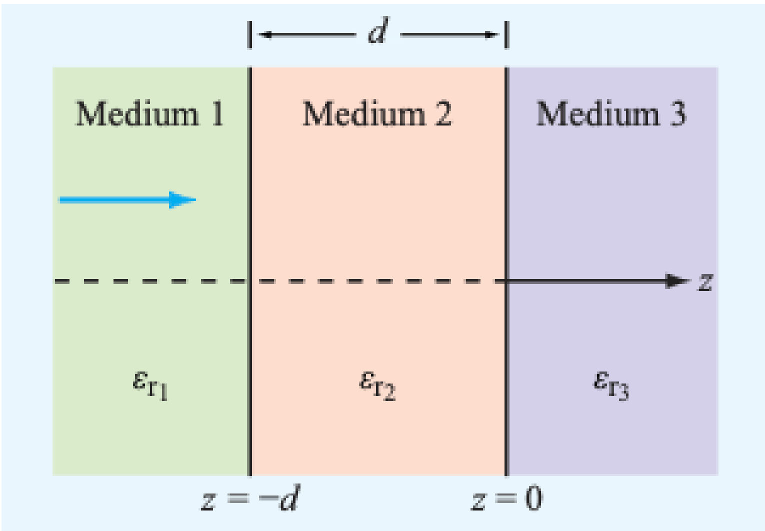
d) Determine the percentages of the incident average power reflected by the boundary and transmitted into the second medium

$$\%\ of\ reflected\ power = \ 100 \times |\Gamma|^2 = \frac{100}{9} = 11.11\%$$

$$\%\ of\ transmitted\ power = 100 \times |\tau|^2 \frac{\eta_1}{\eta_2} = 100 \times \left(\frac{2}{3}\right)^2 \times \frac{120\pi}{60\pi} = 88.89\%$$

Example 6

3 perfect dielectrics (no loss)

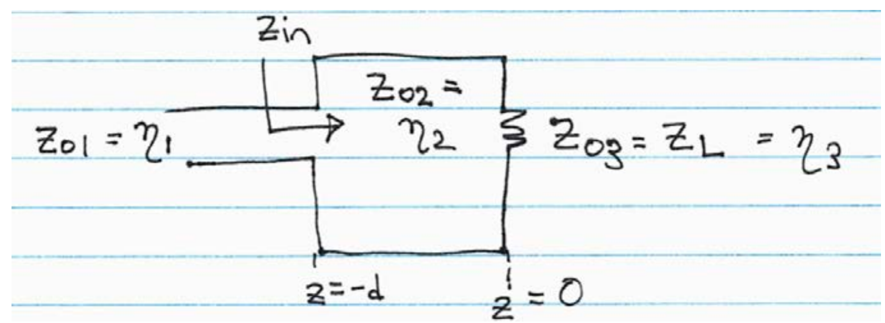


From medium 1

Normal incident wave on medium 2 at z = -d

Find ϵ_{r_2} and d such that no reflection.

Express in terms of ϵ_{r_1}, ϵ_{r_3} and f



Want $Z_{in} = Z_{01} \rightarrow matched$

We have no reflections if medium 2 is $\frac{\lambda}{4}$ transformer:

$\frac{\lambda}{4}$ transformer: $Z_{in} = \frac{Z_{02}^2}{Z_L} = \frac{\eta_2^2}{\eta_3}$

$$l = \frac{\lambda}{4} + \frac{n\lambda}{2}$$

Z_{03}

Example 6

$$\eta = \frac{\eta_0}{\sqrt{\epsilon_r}} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$\eta_1 = \frac{\eta_0}{\sqrt{\epsilon_{r_1}}} \qquad \eta_2 = \frac{\eta_0}{\sqrt{\epsilon_{r_2}}} \qquad \eta_3 = \frac{\eta_0}{\sqrt{\epsilon_{r_3}}}$$

$$\eta_1 = Z_{in} = \frac{\eta_2^2}{\eta_3} \rightarrow \frac{1}{\sqrt{\epsilon_{r_1}}} = \frac{\sqrt{\epsilon_{r_3}}}{\epsilon_{r_2}} \qquad \boxed{\epsilon_{r_2} = \sqrt{\epsilon_{r_1}\epsilon_{r_3}}}$$

$$d = \frac{\lambda_2}{4} + \frac{n\lambda_2}{2}$$ quarter wave → $$\lambda_2 = \frac{\lambda_0}{\sqrt{\epsilon_{r_2}}} = \frac{c}{f}\frac{1}{\sqrt{\epsilon_{r_2}}}$$

$$\lambda_2 = \frac{c}{f}\frac{1}{\left(\epsilon_{r_1}\epsilon_{r_3}\right)^{1/4}}$$

$$d = \frac{c}{4f}\frac{1}{\left(\epsilon_{r_1}\epsilon_{r_3}\right)^{1/4}} + n\left(\frac{c}{2f}\frac{1}{\left(\epsilon_{r_1}\epsilon_{r_3}\right)^{1/4}}\right)$$